GREEN FUNCTION FOR DELTA POTENTIAL
AND
ONE LOOP CORRECTION TO QUANTUM ELECTRODYNAMIC SCATTERING PROCESS

A project report for partial fulfillment of requirements for degree of
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Submitted by

Raj Kishore
410PH5084

Supervised by

Dr. Biplab Ganguli
Department of Physics and Astronomy
National Institute of Technology Rourkela

Abstract

Greens function a technique used to solve in general non homogeneous differential equations. It is basically a correlation function. Its application in high energy physics in finding the propagators has discussed here. In this paper, its application in quantum physics to find Greens functions for quantum operators and its solutions has the main focus. By knowing the Greens function we can calculate density of states. In this paper, contains a detail calculations to find the Greens function for single and double delta function potential and then analysis of the bound state.
In particle physics one of the main observable concern for the experiment is the scattering cross-section. Physicists used the perturbation theory to approximate results of the scattering process, as we do not have the exact knowledge of scattering process. In the higher order corrections, due to loops, the amplitude expression for the Feynman diagram diverges. We apply a theory of renormalization to get a physically possible results. This paper start with the basic introduction to renormalization theory then its application to quantum electrodynamics scattering process. I have briefly discussed all three types of loops found in QED and focused mainly on the vacuum polarization (photon self energy) corrections which give the charge renormalization.

Certificate

This is to certify that the work done in thesis entitled, **“Green’s function for delta potential” and “one loop correction to quantum electrodynamic scattering process”,** are submittedby **Raj Kishore** towards partial fulfillment of the requirements for the award of Master of Science in Physics degree by National Institute of Technology Rourkela. It is a record of the work done by him, under my supervision. The published results have been partly reproduced.

Dr. Biplab Ganguli

Department of Physics and Astronomy

National Institute of Technology Rourkela

India

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Raj Kishore
National Institute of Technology, Rourkela, India
Email:rajkishoreokey@gmail.com

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PART-I

Chapter 1

Definition of Green's function

For any arbitrary linear differential operator $\mathbf{L}$, in $\mathbb{R}^n$ (Euclidean space), the Green's function $G(r, r')$ is defined by the solution of equation

$$\mathbf{L} \cdot G(r, r') = \delta(r_0 - r') \tag{1.1}$$

where, $r, r' \epsilon \mathbb{S}$ where $\mathbb{S}$ is a surface domain and δ in the Dirac delta function. By this definition of green function, we can have the solution of any inhomogeneous differential eqn. of the form

$$\mathbf{L}\phi(r) = f(r) \tag{1.2}$$

and the solution is given by

$$\phi(r) = \int G(r, r') f(r') dr \tag{1.3}$$

Poisson's equation

$$\nabla^2 \psi = \frac{-\rho}{\epsilon_0} \tag{1.4}$$

Now the Green function corresponding to this differential eq. is given by

$$\nabla^2 G(r, r') = +\delta(r_0 - r') \tag{1.5}$$

but from Gauss' theorem

$$\int \nabla^2 (\frac{1}{r} d\tau) = \int \nabla (\frac{1}{r} \cdot d\vec{\sigma}) \tag{1.6}$$

Since,

$$\nabla (\frac{1}{r}) = \frac{-\vec{r}}{r^3} = -\frac{\hat{r}}{r^2} \tag{1.7}$$

So,

$$\begin{aligned} \int \nabla(\frac{1}{r}) \cdot d\vec{\sigma} &= -\int \frac{\hat{r} \cdot d\vec{\sigma}}{r^2} && (1.8) \\ &= \int_0^{4\pi r^2} \frac{\hat{r} \cdot \hat{n} dS}{r^2} && (1.9) \\ &= -\frac{1}{r^2} \int_0^{4\pi r^2} dS && (1.10) \end{aligned}$$

So,

$$\int \nabla^2 (\frac{1}{r}) d\tau = \begin{cases} 0 & \text{if } if\ origin\ is\ excluded \\ -4\pi & \text{if } origin\ is\ included \end{cases} \tag{1.11}$$

or

$$\nabla^2 \frac{1}{r} = -4\pi\delta(r) \tag{1.12}$$

Hence,

$$\nabla^2 (\frac{1}{|r - r'|}) = -4\pi\delta(r - r') \tag{1.13}$$

Now, from eqn (1.5) and (1.13)

$$G(r, r') = -\frac{1}{4\pi|r - r'|} \tag{1.14}$$

So, the solution of eqn. (1.4) can be

$$\psi(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r')}{|r - r'|} dr \tag{1.15}$$

Chapter 2

Green's function in Quantum Physics

The time independent Gree's function can be defined as the solution of inhomogeneous differential eqn of

$$(z - \mathcal{C}(r))G(r, r'; z) = \delta(r - r') \tag{2.1}$$

on some boundary condition for r and r', and $z = \lambda + \iota s$ and $\mathbf{L}(r)$ is a time independent linear Hermitian differential operator.
Let $\phi_n(r)$ be the complete set of eigen functions of the differential operator $\mathbf{L}(r)$ then

$$\mathcal{C}(r)\phi_n(r) = \lambda_n \phi_n(r) \tag{2.2}$$

and $\phi_n(r)$ also satisfy the same boundary conditions as $G(r, r'; z)$. Since, eigen functions ϕ_n corresponding to difficulty eigen values are orthogonal.
So, $\phi_n(r)$ can be considered as orthogonal. So,

$$\int \phi_n^*(r)\phi_m(r)dr = \delta_{nm} \tag{2.3}$$

from completeness condition

$$\sum_n |\phi_n\rangle \langle\phi_n| = 1 \tag{2.4}$$

and

$$\begin{aligned}
\sum_n \phi_n(r)\phi^*(r') &= \sum_n \langle r|\phi_n\rangle \langle\phi_n|r'\rangle && (2.5)\\
&= \sum_n |\phi_n\rangle \langle\phi_n| \langle r|r'\rangle && (2.6)\\
&= \langle r|r'\rangle && (2.7)\\
&= \delta(r - r') && (2.8)\\
& && (2.9)
\end{aligned}$$

and

$$\begin{aligned}
G(r, r'; z) &\equiv \langle r| G(z) |r'\rangle && (2.10)\\
\delta(r - r')\mathbf{L} &\equiv \langle r| \mathbf{L} |r'\rangle && (2.11)\\
\langle r|r'\rangle &\equiv \delta(r - r') && (2.12)
\end{aligned}$$

where $|r\rangle$ is the eigenvector of position operator. In new notations, the eqn. (1.1) can be written as

$$(z - \mathbf{L})G(z) = \mathbf{1} \tag{2.13}$$

Now,
if all eigenvalues of $z - \mathbf{L}$ are non-zero i.e. $z \neq \lambda_n$ then,

$$G(z) = \frac{1}{z - \mathbf{L}} \tag{2.14}$$

$$G(z) = \frac{\sum_n |\phi_n\rangle |\phi_n\rangle}{z - L} \tag{2.15}$$

but eigenvalues of $\mathbf{L}$ can have both discrete and continuous. So,

$$\sum_n = \Sigma' + \int dc \tag{2.16}$$

$$G(z) = \Sigma'_n \frac{|\phi_n\rangle \langle\phi_n|}{z - \lambda_n} + \int dc \frac{|\phi_c\rangle |\phi_c\rangle}{z - \lambda_c}$$

Chapter 3

Green's Functions For Single Delta Function Potentials

Green's function $G(x, y; E)$ associated with the Hamiltonian H is the solution to the eqn.

$$(E - H)G(x, y; E) = \delta(x - y) \tag{3.1}$$

Satisfying the boundary conditions

$$\lim_{|x-y| \to \infty} G(x, y; E) = 0$$

here, x and y are points in D-dimensional Euclidean space and correspondingly $\delta(x - y)$ is a D-dimensional delta function.
Let us suppose $\{\psi_n(x)\}$ be the eigenstates of H
So, $G(x, y; E) = \sum\limits_n \frac{\psi_n(x)\psi_n^*(y)}{E - E_n}$
Let

$$H = H_0 + \lambda\delta(x)$$

Where H_0 is the Hamiltonian of free particle. so, the Hamiltonian H contains one delta potential.
Now, we know the Green's function associated with H_0 i.e. for the free particle.
Let $G_0(x, y; E)$ be the Green's function associated with H_0.
Then,

$$G_0(E) = (E - H_0)^{-1} = \frac{1}{E - H_0}$$

Now,

$$\begin{aligned}
G(E) &= \frac{1}{E - H} && (3.2)\\
&= (E - H_0 - \lambda\delta(x))^{-1} && (3.3)\\
&= G_0(E)\{1 - G_0(E)\lambda\delta(x)\}^{-1} && (3.4)\\
&= G_0(E) + G_0(E)\lambda\delta(x)(G_0(E) + G_0(E)\lambda\delta(x)G_0(E) + E + \dots) && (3.5)
\end{aligned}$$

$$G(E) = G_0(E) + G_0(E)\lambda\delta(x)G(E)$$

In x, y representation,

$$G(x, y; E) = G_0(x, y; E) + \int d^D z G_0(x, z; E)\lambda\delta(z)xG(z, y; E)$$

$$G(x,y;E) = G_0(x,y;E) + \lambda G_0(x,0;E)G(0,y;E)$$

Putting $x = 0$ is in the above eqn.

$$G(0,y;E) = G_0(0.y;E) + \lambda G_0(0,0;E)G(0,y;E)$$

$$G(0,y;E) = \frac{G_0(0,y;E)}{1 - \lambda G_0(0,y;E)}$$

now,

$$G(x,y;E) = G_0(x,y;E) + \frac{\lambda G_0(x,0;E)G_0(0,y;E)}{1 - \lambda G_0(0,0;E)}$$

$$G(x,y;E) = G_0(x,y;E) + \frac{G_0(x,0;E)G_0(0,y;E)}{\frac{1}{\lambda} - G_0(0,0;E)}$$

This is the Green's function associated with Hamiltonian H.

3.1 Bound States

Bound states of the Hamiltonian H with H_0, the Hamiltonian of the free particle.
We know that poles of the Greens function gives the eigenvalues of the eigenvalues of the Hamiltonian.
So, poles of $G(x,y;E)$ in the above eqn. are the energy levels in bound states.
Since, there are no bound state in free particle problem, so from the above eqn. only poles occur if

$$\frac{1}{\lambda} - G_0(0,0;E) = 0$$

Suppose, $H_0 = -\nabla^2$ then,

$$G_00(x,y;E) = \int \frac{d^Dk}{(2\pi)^D} \frac{e^{\iota k}(x-y)}{E - k^2}$$

So, in order to find the energy of bound states, we must solve the equation

$$\frac{1}{\lambda} + \int \frac{d^Dk}{(2\pi)^D} \frac{1}{K^2 + k^2} = 0 \tag{3.6}$$

(where $k^2 = -E$) now, for 1-dimension

$$\int_{-\infty}^{\infty} \frac{dk}{(2\pi)} \frac{1}{K^2 + k^2}$$

Let $k = K \tan\theta$, then

$$dk = K \sec^2\theta d\theta$$

$$\int_{-\infty}^{\infty} \frac{dk}{K^2 + k^2} = \frac{1}{K} \int_{-\pi/2}^{\pi/2} d\theta = \frac{\pi}{K}, k > 0$$

So,

$$\int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{dk}{K^2 + k^2} = \frac{1}{2K}$$

This means,

$$\frac{1}{2k} + \frac{1}{\lambda} = 0$$

which gives,

$$k = -\frac{\lambda}{2}$$

$$E_B = -\frac{\lambda^2}{4}$$

since, $k > 0$, so, $\lambda < 0$.
So, physically, this means that the potential must be attractive in order to create a bound state.
This result is also consistent with the more elementary derivation. The time-independent schrodinger eqn. for a particle in potential

$$V(x) = \lambda\delta(x)$$

is

$$-\frac{d^2}{dx^2}\psi(x) + \lambda\delta(x)\psi(x) = E\psi(x) \tag{3.7}$$

for $x \neq 0$
the eqn. becomes

$$-\frac{d^2}{dx^2}\psi(x) = E\psi(x)$$

$$\frac{d^2}{dx^2}\psi(x) = k^2\psi(x)\{k^2 = -E\}$$

$$\psi(x) = Ae^{-k|x|}$$

$$\psi(x) = \begin{cases} Ae^{-kx} & \text{if } x > 0 \\ Ae^{kx} & \text{if } x < 0 \end{cases}$$

by using boundary condition at $x = 0$. We have to integrate eqn (3.7) in the interval $(-\epsilon, \epsilon), \epsilon > 0$

$$-\int_{-\epsilon}^{\epsilon} \frac{d^2}{dx^2} + \int_{-\epsilon}^{\epsilon} \lambda\delta(x)\psi(x) = \int_{-\epsilon}^{\epsilon} E\psi(x) \tag{3.8}$$

$$-\psi'(0^+) + \psi'(0-) + \lambda\psi(0) = 0$$

This gives us

$$k = -\frac{\lambda}{2}$$

Chapter 4

Green's Function For Double Delta Function Potential

The Hamiltonian of the double delta function potential can be given as;

$$H = H_0 + \lambda\delta(x) + \lambda\delta(x-a) \tag{4.1}$$

we have the Green' function for single delta function potential;

$$H = H^1 + \lambda\delta(x-a) \tag{4.2}$$

$$H^1 = H_0 + \lambda\delta(x) \tag{4.3}$$

The Green's functions for the Hamiltonian H can be written as:

$$G(E) = \frac{1}{E-H} \tag{4.4}$$

$$G(E) = [E - H_0 - \lambda\delta(x) - \lambda\delta(x-a)]^{-1} \tag{4.5}$$

$$G(E) = (E-H_1)^{-1}[1 - G^1(E)\lambda\delta(x-a)]^{-1} \tag{4.6}$$

$$G(E) = G^1(E)[[1 - G^1(E)\lambda\delta(x-a)]^{-1} \tag{4.7}$$

Now, expanding this in the power series,
it reduces to:

$$G(E) = G^1(E) + G^1(E)\lambda\delta(x-a)G(E) \tag{4.8}$$

In the x,y representation,

$$G(x,y;E) = G^1(x,y;E) + \int d^D G^1(x,z;E)\lambda\delta(z-a)G(z,y;E) \tag{4.9}$$

where x,y are any two poins in D-dimension space.

For finding the Green's function explicitly
Let, x=a put it in equation(8) then the equation gives;

$$G(a,y;E) = \frac{G^1(a,y;E)}{1 - \lambda G(a,a;E)}$$

then,

$$G(x,y;E) = G^1(x,y;E) + \frac{G^1(a,y;E)G^1(x,a;E)}{1/\lambda - G^1(a,a;E)}$$

$$G(x,y) = \frac{[G_0(a,y) + \frac{G_0(a,0)G(0,y)}{1/\lambda - G_0(0,0)}][G_0(x,a) + \frac{G_0(x,0)G_0(0,y)}{1/\lambda - G_0(0,0)}]}{1/\lambda - G_0(a,a) - \frac{G_0(a,0)G_0(0,a)}{1/\lambda - G_0(0,0)}} + G^1(x,y) \tag{4.10}$$

But from this expression we can say that the point of singularity for $G_0(x,y)$ is also the singularity for $G(x,y)$.

Since,

$$G(x,y) = G^1(x,y) + \frac{G^1(x,a)G^1(a,y)}{1/\lambda - G^1(a,a)}$$

from this expression, since, G^1 already has a bounded state;

$$E_1 = -\lambda^2/4$$

To have another bounded state from the above equation

$$\frac{1}{\lambda} - G^1(a,a) = 0$$

$$\frac{1}{\lambda} - G_0(a,a) - \frac{G_0(a,0)G_0(0,a)}{1/\lambda - G_0(0,0)} \tag{4.11}$$

$$(\frac{1}{\lambda})^2 - \frac{2}{\lambda}G_0(0,0) + (G_0(0,0))^2 - G_0(a,0)G_0(0,a) = 0 \tag{4.12}$$

Since,

$$G(x,y;E) = \int \frac{d^Dk}{(2\pi)^D}\frac{e^{\iota k.(x-y)}}{E - K^2} \tag{4.13}$$

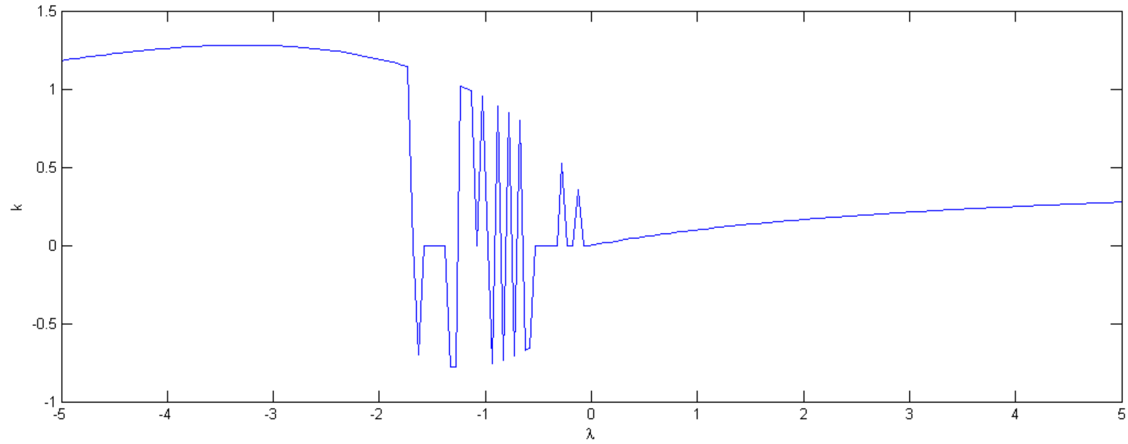
From this we calculate;

$$G_0(0,0) = \frac{-1}{2K}$$

$$G_0(a,0)G_0(0,a) = \frac{1}{8K^3}$$

Putting all this in the above equation, we obtain :

$$\boxed{K^3 + \lambda K^2 + \frac{\lambda^2}{4}K - \frac{\lambda^2}{8} = 0} \tag{4.14}$$



From the graph of K vs λ, for any positive value of $\lambda > 0$ there is a value of K, so there exist a bound state.

For $\lambda < 0$, there are many bumps in the graphs but since the graph is continuous, one thing we can conclude that there exists values for K for a given value of λ. So there must exist bound state.
For the larger values of λ, as from the graph, there can exist more than one bound states.

4.1 Conclusion

[1] For $\lambda > 0$ there is a value of K, so there exist a bound state.

[2] For $\lambda < 0$, there are many bumps in the graphs but since the graph is continuous, one thing we can conclude that there exists values for K for a given value of λ. So there must exist bound state.

[3] For the larger values of λ, as from the graph, there can exist more than one bound states.

PART-II

Chapter 5

Some Necessary Mathematics Notations and Conventions

Units
We have followed the "God-given" units called natural units, where;

$$\hbar = c = 1$$

In this system:

$$[length] = [time] = [energy]^{-1} = [mass]^{-1}$$

Four-Vectors
An example of a four-vector is the space-time coordinates(t, x, y, z).
In general, any set of four quantities which transform like $(t, \mathbf{X})$ under Lorentz transformation is $four - vector$.
The basic invariant under Lorentz transformation is $(t^2 - \mathbf{X}^2)$

$$(t, \mathbf{X}) \equiv (x^0, x^1, x^2, x^3) \equiv x^\mu$$

According to special relativity, total energy E and the momentum $\mathbf{p}$ of an isolated system transform as the component of $four - vector$.

$$(E, \mathbf{p}) \equiv (p^0, p^1, p^2, p^3) \equiv p^\mu$$

Lorentz Transformation:

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1) \\ x'^1 &= \gamma(x^1 - \beta x^0) \\ x'^2 &= x^2 \\ x'^3 &= x^3 \end{aligned}$$

Lorentz Invariant:
We have already seen that the quantity

$$t^2 - x^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

is invariant under Lorentz transformation.
In general invariant can be contracted from any pair of four-vector. For example, let k and p are four-momenta then its *scalar product*;

$$p \cdot k = p^0 k^0 - p^1 k^1 - p^2 k^2 - p^3 k^3$$

is invariant.
Just in $3-D$ space, we may introduce the scalar product of two four-vectors $A^\mu \equiv (A^0, \mathbf{A})$ and $B^\mu \equiv (B^0, \mathbf{B})$then;

$$A \cdot B \equiv A^0 B^0 - A^1 B^1$$

another four-vector $A_\mu \equiv (A^0, -\mathbf{A})$so, the scalar product;

$$A \cdot B = A_\mu B^\mu = A^\mu B_\mu = g_{\mu\nu} A^\mu B^\nu = g^{\mu\nu} A_\mu B_\nu$$

where $g_{\mu\nu}$ or $g^{\mu\nu}$ are metric tensor given by;

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Chapter 6

Baseline Concepts

Chapter 7

Relativistic Quantum Mechanics

7.1 Klein-Gordon Equation

The relativistic energy-momentum relation:

$$E = \sqrt{\mathbf{p}^2c^2 + m^2c^4}$$

in natural units $\hbar = c = 1$

$$E = \sqrt{\mathbf{p}^2 + m^2}$$

so,

$$\iota\frac{\partial\psi}{\partial t} = \sqrt{\mathbf{p}^2 + m^2}\psi$$

this equation is undesirable because it treats space and time asymmetrically.But what we want is an equation that is of the same order in both space and time.This can be done in two ways;
1st way: replace equation $E = \sqrt{\mathbf{p}^2 + m^2}$ by $E^2 = \mathbf{p}^2 + m^2$ so the wave equation become

$$\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi + m^2\psi = 0 \tag{7.1}$$

this is the Klein-Gordon equation. in D' Alembertian operator form: From the Klein-Gordon equation, probability and flux densities form a four- vector;

$$j^\mu = (\rho, \mathbf{j}) = \iota(\psi^*\partial^\mu\psi - \psi\partial\psi^*) \tag{7.2}$$

which satisfies the (covariant) continuity relation

$$\partial_\mu j^\mu = 0 \tag{7.3}$$

7.2 Dirac Equation

The general form of Dirac equation;

$$H\psi = (\alpha \cdot \mathbf{p} + \beta m)\psi \tag{7.4}$$

The four coefficients β and $\alpha_i (i = 1, 2, 3)$ are represented by:
Dirac-Pauli representation:

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

where σ are Pauli-matrices. I denotes the $2*2$ unit matrix.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Wely-representation;

$$\alpha = \begin{pmatrix} -\sigma & 0 \\ 0 & \sigma \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

Covariant form of the Dirac Equation

The equation:

$$(\iota\gamma^\mu\partial_\mu - m)\psi = 0 \tag{7.5}$$

where γ^μ is four Dirac $\gamma - matrices$

$$\gamma^\mu \equiv (\beta, \beta\alpha)$$

Properties of Dirac $\gamma - matrices$

Dirac $\gamma - matricse$ satisfy the anti-commutation relation.

$$\gamma^\mu\gamma^\nu + \gamma^{nu}\gamma^{mu} = 2g^{\mu\nu}$$

$$\begin{cases} (\gamma^k)^\dagger = -\gamma^k \\ (\gamma^k)^2 = -I \qquad where\ k = 1,2,3 \end{cases}$$

$$(\gamma^0)^\dagger = \gamma^0 \qquad (\gamma^0)^2 = I$$

Note that the hermitian conjugation result can be summarized by;

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$$

Conserved current and adjoint equation

The adjoint equation of the covariant Dirac equation is;

$$\iota\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0 \tag{7.6}$$

where $\bar{\psi} \equiv \psi^\dagger\gamma^0$ is called the *adjoint row spinor*.
to derive the continuity equation $\partial_\mu j^\mu = 0$,multiply$eq^n 8$ from left by $\bar{\psi}$ and $eq^n 9$ from right by ψ and then on addition we will get;

$$\partial_\mu(\bar{\psi}\gamma^\mu\psi = 0$$

so,

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

now,

$$\rho = j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi = |\psi|^2$$

so the probability density is positive definite.
From the Pauli-Weisskopf representation that j^μ should be identified with the charge current density. we therefore insert charge $-e$ in j^μ,

$$j^\mu = -e\bar{\psi}\gamma^\mu\psi \tag{7.7}$$

So, j^μ is the electron $(four - vector)$ current density.

Chapter 8

Scattering Amplitude Calculation from Feynman Diagram

We can get the scattering amplitude from the non-relativistic perturbation theory. Lets suppose the known Solution to the free- particle Schrodinger equation be:

$$H_0\phi_n = E_n\phi_n \qquad with \qquad \int_V \phi_m^*\phi_n d^3x = \delta_{mn}$$

where H_0, Hamiltonian ,is time independent. For time dependent perturbation $V(x,t)$, Schrodinger equation:

$$(H_0 + V(x,t))\psi = \iota\frac{\partial\psi}{\partial t}$$

any solution can be expressed in form

$$\psi = \sum_n a_n(t)\phi_n(x)e^{-\iota E_n t}$$

then the coupled linear differential equation for the a_n coefficients:

$$\frac{\partial a_f}{\partial t} = -\iota\sum_n a_n(t)\int \phi_f^* V\phi_n d^3x e^{\iota(E_f-E_n)}t$$

now let;

$$\begin{cases} a_i(-T/2) = 1 \\ a_n(-T/2) = 0 \end{cases}$$

and then

$$\frac{\partial a_f}{\partial t} = -\iota\int \phi_f^* V\phi_i d^3x e^{\iota(E_f-E_i)}t$$

assume these initial conditions remain true at all times. Then on integration, we obtain

$$a_f(t) = -\iota\int_{-T/2}^{t} dt'\int \phi_f^* V\phi_i d^3x e^{\iota(E_f-E_i)}t$$

and in particular at time $t = +T/2$ after the interaction has ceased,

$$T_{fi} \equiv a_f(T/2) = -\iota\int_{-T/2}^{+T/2} dt\int d^3x[\phi_f(x)e^{-\iota E_f t}]^* V(x,t)[\phi_i(x)e^{-\iota E_i t}$$

in covariant form

$$T_{fi} = -\iota \int d^4x \phi_f(x) V(x) \phi_i(x) \tag{8.1}$$

This T_{fi} is the transition amplitude.

This is only the 1st order correction.
For higher order corrections, use the result of 1st order correction; so,

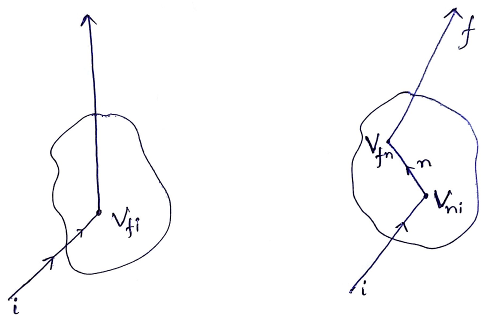
$$\frac{da_f}{dt} = ... + (-\iota)^2 [\sum_{n\neq} V_{ni} \int_{-T/2}^{t} dt' e^{\iota(E_n - E_i)t}] V_{fn} e^{\iota(E_f - F_i)} \tag{8.2}$$

where the dots represents the 1st order corrections.
So, the corrections to T_{fi} is;

$$T_{fi} = ... - \sum_{n\neq i} V_{fn} V_{ni} \int_{-\infty}^{\infty} dt e^{\iota(E_f - E_n)t} \int_{-\infty}^{t} dt' e^{\iota(E_n - E_i)t'} \tag{8.3}$$

The final expression for the scattering amplitude T_{fi} upto the 2nd order corrections is :

$$T_{fi} = -2\pi\iota V_{fi} \delta(E_f - E_i) - 2\pi\iota \sum_{n\neq i} \frac{V_{fn} V_{ni}}{E_i - E_n + \iota\epsilon} \delta(E_f - E_i) \tag{8.4}$$



Propagator

A propagator is basically related to the Green's function for the operator.
Photon propagator come into picture when there is any electromagnetic interactions. When two particles (A and B) with momenta p_A and p_B qet interact via electromagnetic interaction then the propagator for this field is basically the photon field and it is given by

$$Photon \quad Propagator = \frac{-\iota g_{\mu\nu}}{(p_A + p_B)^2}$$

$$Electron \quad propagator = \frac{\iota(\not{p} + m)}{(p^2 - m^2)}$$

$$Spinless \quad particle = \frac{\iota}{(p^2 - m^2)}$$

Introduction

Renormalization is a mathematical concept (or a way) by which we can get rid of infinities incountering in many physical phenomena at high energy or equivalently say penomena at very short distances.
In the quantum field theory (QFT) and particle physics we encounter some divergences which are called **ultraviolet divergences**, which usually arises by the contribution of very high energy term or the terms for physical phenomena at very short distances.
Ultraviolet divergences were 1st found in QED. To eliminate the divergences, many physicists believed that fundamental principles of physics had to be changed. In the 1940's Feynman, Bethe, Tomonaga, Schwinger and Dyson, and others proposed a theory of *renormalization* that gave the physically sensible results by redefining the physical quantities to absorb the divergences. That gave the most precise calculations which matches with experiment to 8 significant digits in QED, which was the most accurate calculations in all of science.
The modern understanding of renormalization was laid by K. Wilson in the 1970's. According to the present view, renormalization is nothing more than parameterizing the sensitivity of low-energy physics to high-energy physics.

In particle physics one of the main observable concern for the experiment is the scattering cross-section, which is the probability of interaction between elementary particles. Quantum mechanics tells that such probabilities are calculated as squares of complex scattering amplitudes. Unfortunately we do not have the knowledge of even the simplest scattering amplitude exactly. Physicists used the perturbation theory to approximate results of the scattering process.
Perturbation theory uses a beautiful tool known as Feynman diagrams, a graphic visualization of terms in the series in powers of the coupling strength between interacting particles. Thee level diagram is the 1st order contribution to scattering amplitudes. and it is the largest one. In the higher order corrections there are loops in the diagram.
The 1st order or the tree level computation is well established, but the higher order diagrams contains loops whose computations are very challenging. Due to loops, the amplitude expression for the Feynman diagram diverges, which physicists apply a theory of renormalization to get a physically possible results. It was first applied to the quantum electrodynamics processes and shows excellent results. The anomalous magnetic moment of the electron has been correctly predicted by this renormalization theory which is one of the great successes of perturbation theory.

Chapter 9

Renormalization in Quantum mechanics

Field theory is not the only theory, but also in some simple quantum mechanical models, one can find ultraviolet divergences and then there will be a need for renormalization.

9.1 In One -Dimension

In one -dimensional quantum mechanics, Hamiltonian described by

$$\hat{H} = \frac{1}{2}\hat{P} + \hat{V} \tag{9.1}$$

here we have used the natural units: $\hbar = 1$ and $c = 1$. In these units all quantities have dimensions of the length to some power. Here also we have taken mass $m = 1$.

$$[P] = \frac{1}{L} \qquad [E] = \frac{1}{L^2}$$

Lets suppose that a potential $V(x)$ has the center origin and a range of order a and a height of order V_0 Now, we are going to look for the scattering of incoming particle wavefunction from a local potential in one-dimensional quantum mechanics.

local.png

so, lets suppose an incident plane wave with momentum $p > 0$ to the left of $x = 0$. That is, we assume the asymptotic forms of the position-space wavefunction.

$$\psi(x) = \begin{cases} Ae^{\iota px} + Be^{-\iota px} & x \to -\infty \\ Ce^{\iota px} & x \to \infty \end{cases} \tag{9.2}$$

where the contributions proportional to A C and B represent the incoming, transmitted and the reflected waves respectively.
Now, Suppose that the range of the potential a is small compared to the de Broglie wavelength $\lambda = 2\pi/p$,(since,$\hbar = 1$). It means that the incoming wavefunction to the left of $x = 0$ is approximately constant over the range of the potential, and we can expect

that the details of the potential is not needed. We can then approximate(to a good approximation) the potential by a delta function :

$$V(x) = c\delta(x) \tag{9.3}$$

where c is a phenomenological parameter ('coupling constant'), and the dimension[1] of c is

$$[c] = \frac{1}{L}$$

Now, by approximating the potential by a delta function, it can be justified by considering trial wavefunctions $\psi(x)$ and $\chi(x)$ that vary on a length scale $\lambda >> a$.
Consider matrix elements of the potential between such states:

$$< \chi|\hat{V}|\psi >= \int dx \chi^*(x) V(x) \psi(x)$$

Since the wavefunctions $\chi(x)$ and $\psi(x)$ are approximately constant in the region where the potential is non-vanishing, so we can write

$$< \chi|V\hat{V}|\psi > \approx \chi^*(0)\psi(0) \int dx V(x)$$

This is equivalent to the approximation $Eq.$(2.3) with

$$c = \int dx V(x)$$

Now, solution of the time independent Schrodinger equation:

$$-\frac{1}{2}\psi^{''}(x) + c_0\delta(x)\psi(x) = E\psi(x) \tag{9.4}$$

can be written as

$$\psi(x) = \begin{cases} Ae^{\iota px} + Be^{-\iota px} & x < 0 \\ Ce^{\iota px} & x > 0 \end{cases} \tag{9.5}$$

where $p = \sqrt{2E}$. On integrating the above Schrodinger equation in the interval $(-\epsilon, \epsilon)(\epsilon$ is small +iv constant) containing the origin;

$$\int_{-\epsilon}^{\epsilon} [-\frac{1}{2}\psi^{''}(x) + c_0\delta(x)\psi(x)]dx = E\int_{-\epsilon}^{\epsilon} \psi(x)dx$$

$$-\frac{1}{2}[\psi^{'}(\epsilon) - \psi^{'}(-\epsilon)] + c_0\psi(0) = O(\epsilon)$$

taking $\epsilon \to 0$;

$$\frac{\iota p}{2}(C - A + B) + c_0 C = 0 \tag{9.6}$$

and for $\psi(0)$ to be well defined, the wavefunction to be continuous at $x = 0$, which gives;

$$A + B = C \tag{9.7}$$

[1]note that the $\delta(x)$ has the unites of $1/L$, (since $\int \delta(x)dx = 1$)

on solving these equations, we have

$$Transmission \quad Amplitude \quad T = \frac{C}{A} = \frac{p}{p + \iota c_0}$$

and

$$Reflection \quad Amplitude \quad R = \frac{B}{A} = -\frac{\iota c_0}{p + \iota c_0}$$

and hence

$$|R|^2 + |T|^2 = 1 \qquad (conservation \quad of \quad Probability)$$

from this, we expect this to be an accurate result for any short range potential as long as $p << 1/a$. So, we have

$$T \approx -\frac{\iota p}{c_0} \tag{9.8}$$

and this is also consisted with the dimensional analysis as T is dimensionless.
This $eq.2.8$ is the "*Low-Energy Theorem*" for the scattering from a short- range potential in the one-dimensional quantum mechanics.
Now if, however, that the short-range potential is an odd function of x:

$$V(-x) = -V(x)$$

then, the 1st non zero term in the Taylor expansion of $V(x)$ is

$$V(x) \approx c_1 \delta(x)$$

then the Schrodinger equation is

$$-\frac{1}{2}\psi^{''}(x) + c_1\delta^{'}(x)\psi(x) = E\psi(x) \tag{9.9}$$

this equation do not have any solution as the 1st derivative of wavefunction $\psi^{'}(x)$ is coming out to be discontinuous at $x = 0$, this can be seen from the jump condition;

$$\int_{-\epsilon}^{\epsilon} [-\frac{1}{2}\psi^{''}(x) + c_1\delta^{'}(x)\psi(x)]dx = E\int_{-\epsilon}^{\epsilon} \psi(x)dx$$

$$-\frac{1}{2}[\psi^{'}(\epsilon) - \psi^{'}(-\epsilon)] + c_1\psi^{'}(0) = O(\epsilon)$$

The problem is that $\psi^{'}(0)$ is not well-defined, because the jump condition tells us that $\psi^{'}$ is discontinuous at $x = 0$. This inconsistency signifies an ultraviolet divergence, which we see later that this ultraviolet divergence is precisely analogous to the one we find in quantum field theory that due to sensitivity to high-momentum modes.

We can conclude from this that a phenomenological description is simply not possible beyond the delta function approximation. If the universal low-energy theorems, such as $Eq.(2.8)$, were not governing the low-energy behavior, then it would mean that the detailed informations about physical phenomena at arbitrarily short distances can be obtained from measurements at long distances. But this counters to our experiences and intuition that the short-distance features of physical systems cannot be solved by the experiments at low energies . So, we therefore expect that the low-energy theorems governs low-energy behavior of the physical systems.

For the scattering from a short-range potential of this form, if there exist an universal low energy form, we would work it out with an arbitrary odd short-range potential. We will use

$$V(x) = c_1 \frac{\delta(x+a) - \delta(x-a)}{2a}$$

As $a \to 0, V(x) \to c_1 \delta^{'}(x)$, so this can be viewed as a discretization of the potential. For $a \neq 0$, we have a well-defined potential with width of order a. The parameter a is a *short-distance cutoff*.

We can calculate the transmission amplitude C by writing a solution of the form;

$$\psi(x) = \begin{cases} Ae^{\iota px} + Be^{-\iota px} & x < -a \\ A'e^{\iota px} + B'e^{-\iota px} & -a < x < a \\ Ce^{\iota px} & x > a \end{cases}$$

by applying continuity and the jump conditions at $x = a$ and $x = a$. The solution we have is;

$$\frac{1}{T} = \frac{c_1^2}{4a^2p^2}(1 - e^{\iota 4ap})$$

but it diverges as $a \to 0$. However, for low energy case: $p << a$, we get

$$T = -\frac{\iota ap}{c_1^2}$$

Note that the cutoff theory initially depends on 2 parameters, namely c_1 and the cutoff a. However, this result shows that the low-energy behavior depends only on the combination

$$c_R = \frac{c_1^2}{a}$$

so, the low energy theorem can be written as

$$T \approx -\frac{\iota p}{c_R} \tag{9.10}$$

which depends on the single phenomenological parameter c_R. The final result is independent of the cutoff $'a'$ in the view that changing c_1, can compensate for a change in a. The result in the *eq.*(2.10) is not consisted with the view point of dimensional analysis, as c_1 and c_R have different dimensions. Since, the fact that c_1 is dimensionless, so, by dimensional analysis it would tell us that the dimensionless transition amplitude cannot depends on the momentum. However, because the cutoff parameter a has dimension, the renormalized parameter can have a different dimension than the coupling in the Hamiltonian. We say that the renormalized coupling has an *anomalous dimension*.

So, in conclusion if there is any divergence, we need to take the following steps to get rid of divergences and got some physically allowed solutions.

Ultraviolet divergences

When we do a computation using local interactions(delta functions and their derivatives), we generally find inconsistency results due to a short-distance divergences. The origin of these divergences in the fact that quantum mechanics requires sums over a complete set of states, so quantum corrections in are related sensitively to the properties of high-momentum (in ultraviolet region) states.

Regularizations

We modify the theory at a distance scale of order a (the cutoff) to parameterize the sensitivity to short distance, so that it is well-defined. This we call that the theory has been *regularized*.
In the theory with the cutoff, the ultraviolet divergences are replaced by the sensitivity to a, in the sense that the physical quantity diverges in the limit $a \to 0$ with the fixed coupling.

Renormalization

Apparently there has one more parameter than the original continuum theory in the regulated theory, that is the cutoff. However, when we compute physical quantities, it is find that they actually depend only on a combination of the cutoff and the other parameters. In other words, a change in the cutoff can be absorbed by a change in the couplings so that the invariance of all the physical quantities. We therefore, obtain a well-defined and finite results that depend on the equal number of parameters as the original.

9.2 In two-dimension

Consider a short-range local potential in 2-spatial dimensions. Let the potential to be approximated by a delta function, hence the Schrodinger equation is

$$-\frac{1}{2}\nabla^2\psi + c_0\delta^2(x)\psi(x) = E\psi(x) \tag{9.11}$$

coupling constant c is dimensionless.
Now, to solve this problem we need to regulate the delta function.
For spherically symmetric solutions

$$\nabla^2\psi = \frac{1}{r}\frac{d}{dr}(r\frac{d\psi}{dr}$$

and

$$\delta^2(x) = \frac{1}{2\pi r}\delta(r)$$

so, that $\int d^2(x)\delta^2(x) = 1$. Now, regulating the delta function by replacing

$$\delta(r) \to \delta(r - a) \tag{9.12}$$

where 'a' is a cutoff. then the Schrodinger equation can be

$$-\frac{1}{2r}\frac{d}{dr}(r\frac{d\psi}{dr}) + \frac{c}{2\pi r}\delta(r-a)\psi(r) = E\psi(r) \tag{9.13}$$

For $r \neq a$, the general solution is a linear combination of Bessel function $J_0(pr)$ and $Y_0(pr)$.
Since for small x,

$$J_0(x) = 1 - \frac{x^2}{4} + O(x^4) \tag{9.14}$$

$$Y_0(x) = \frac{2}{\pi}(ln\frac{x}{2} - \gamma)(1 + O(x^2)) \tag{9.15}$$

where $\gamma = 0.577216..$ is the Euler constant. The solution for $r < a$ involves Bessal function that is regular at the origin. So,

$$\psi(r) = \begin{cases} CJ_0(pr) & r < a \\ AJ_0(pr) + BY_0(pr) & r > a \end{cases} \tag{9.16}$$

where $p = \sqrt{2E}$. on applying the boundary conditions,as integrating the $eq.$(2.13) from $a - \epsilon$ to $a + \epsilon$,

$$\int_{a-\epsilon}^{a+\epsilon}[-\frac{1}{2r}\frac{d}{dr}(r\frac{d\psi}{dr}) + \frac{c}{2\pi r}\delta(r-a)\psi(r)]dr = E\int_{a-\epsilon}^{a+\epsilon}\psi(r)dr$$

on taking $\epsilon \to 0$, we get,

$$\psi^{'}(a+\epsilon) - \psi^{'}(a-\epsilon) = \frac{c}{pi}\psi(r) \tag{9.17}$$

For continuity of psi at $r = a$, that gives two equation in A, B and C. on solving for A and B and expanding for $p << 1/a$, we get

$$A = [1 - \frac{c}{pi}(ln\frac{pa}{2} + r) + O(p^2a^2)]C \tag{9.18}$$

$$B = [\frac{c}{2} + O(p^2a^2)]C \tag{9.19}$$

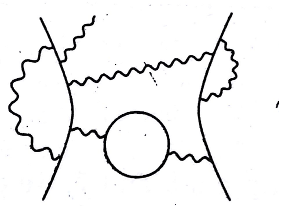
Chapter 10

Ultraviolet Divergences

We have already defined the term "Ultraviolet Divergences" which is generally found in quantum field theory. It is a situation in which an integral or the Feynman diagram diverges because of contribution of the objects with very high energy ($p \to \infty$)or equivalently, because of the physical phenomena at very short distances.
The ultraviolet divergences can be removed by *regularization* and *renormalization.* If a theory can not be renormalized or if the divergences can not be removed, they imply that the theory is not perturbatively well defined at very short distances.
In general, we encounter three quantum electrodynamics diagram(Feynman diagram) with ultraviolet divergences.

10.1 Classification of ultraviolet divergences appear in QFT

Expression corresponding to a typical Feynman diagram can be given as



$$\int \frac{d^4k_1 d^4k_2 ... d^4k_n}{(\not{k}_i - m)....(k_j^2)..(k_n^2)}$$

For each **loop** there is a potentially divergent 4-momentum integral, but each propagator aids the convergence of this integral by putting one or two powers of momentum into the denominator. The diagram diverges unless there are more powers of momentum in the denominator than in the numerator.
So, we can expect the divergence must be proportional to [1]Λ^D when $D > 0$, we expect a divergence of the form $log\Lambda$ when $D = 0$ and no divergence when $D < 0$. Here D is the *superficial degree of divergence*

$$D \equiv \textit{(powers of k in numerator)} - \textit{(powers of k in denominatro)}$$

and D is given by

$$D = 4L - P_e - 2P_\gamma$$

[1]Λ is a momentum cutoff

$$\boxed{D = 4 - N_\gamma - \frac{3}{2} N_e}$$

N_e = number of external electron lines.
N_γ= number of external photon lines.

Chapter 11

Radiative Correction

So far in *Quantum electrodynamics* we have dealt with only first order *Feynman diagram* which is basically called the *tree level process.* It means the diagram does not contain any loop. But all such processes have higher order contributions which are known as *Radiative corrections.*
There is another source of radiative correction in QED is the *Bremsstrahlung* whics is the emission of extra final-state photons during an interaction.

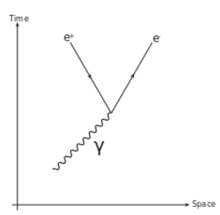
⋆ Why does a loop or many loops can be formed during a process of interaction of two particles or a particle get scattered by a potential?

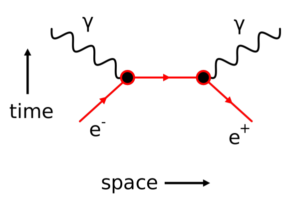
On the very fundamental level, vacuum is actually not a vacuum. There exist *quantum fluctuation*(this is a spontaneous process) at every point of space-time.
In quantum physics, a quantum fluctuation(quantum vacuum fluctuation or vacuum fluctuation) is the temporary change in the amount of energy in a point in space-time as according to Werner Heisenberg's Uncertainty principle.

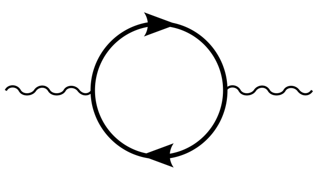
$$\boxed{\Delta E \Delta t \approx \hbar/2}$$

This means that conservation of energy appear to be violated, but only for small time. This allows the creation of particle -antiparticle pairs of virtual particle.
Although these particles are virtual and appear only for small time, its effect can be measured.
In the modern view, energy is always conserved, but the eigenstates of Hamiltonian are not the same as the particle number operator.
In quantum field, fields undergo quantum fluctuations. These quantum fluctuations in turn give rise to *loop formation.*

As the particle-antiparticle gets created due to quantum fluctuation called *pair production* and another phenomenon, reverse of pair production, is pair annihilation in which a pair of particle and antiparticle get annihilated. These combined phenomena is give rise to a loop formation.







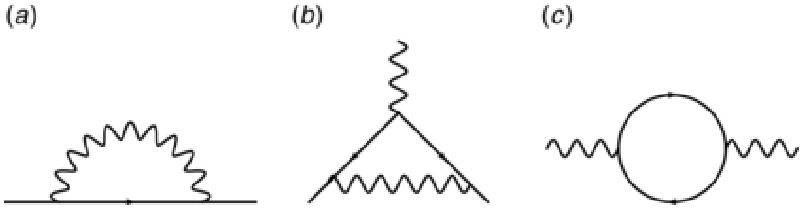
In quantum electrodynamics (QED), we encounter the following four Feynman diagram (three types) at the next higher order in perturbation theory.

[1] *Vacuum polarization*

[2] *Vertex correction*

[3] *External leg correction(self energy)*

The order-α correction to the cross section comes from the interference term between these diagrams and the tree-level diagrams.

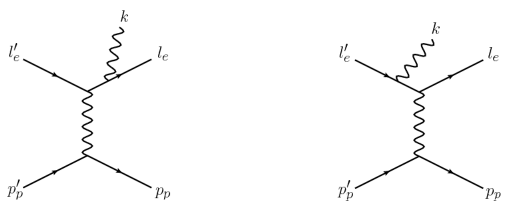


Note: There are six additional one loop diagrams involving the heavy particle in the loop, but they can be neglected in the limit where that particles are much heavier than the electrons. Since, the mass appears in the denominator of the propagator (physically the heavy particles accelerates less, and therefore radiates less, during the collision).

From the four Feynman diagram that are discussed above are one loop diagrams (*radiative correction*). The 1st diagram(a) is known as *external leg corrections*(or the self energy correction). The 2nd diagram(b) is the most intricate(highly involved) and gives the largest variety of new effects called the *vertex correction.* It give rise to an anomalous magnetic moment to the electron.
The last diagram(c) is called the *Vacuum polarization.*

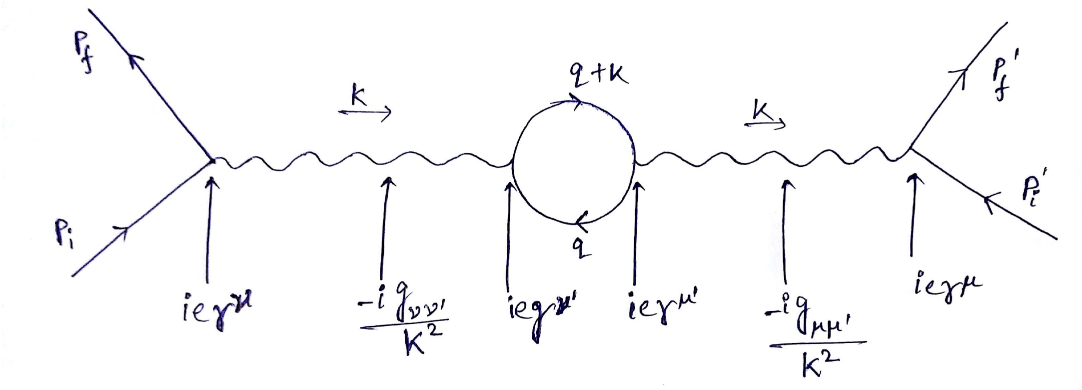
These all corrections are complicate to study as they are ill-defined. Each diagrams involves an integration over the undetermined loop momentum. In each case the integral is divergent in the limit $k \to \infty$ (k is loop momentum)or the *ultraviolet region.*
But, fortunately the infinite parts of these integrals will always cancel out the expressions for the observable quantities such as cross-section.

Apart from the divergence in the ultraviolet region($k \to \infty$), some loops(Vertex correction and the external leg correction) also contains *Infrared divergences* , *i.e.* the infinities coming from the "$k \to 0$" end of the loop-momentum integrals.
These divergences can get canceled when we also include the following *Bremsstrahlung* divergences.



These diagrams diverges in the limit where the energy of the radiated photon tends to zero. In this limit the energy of the photon can not be observed by any physical detector, so it makes sense to add the cross section for producing these low-energy photons to the cross section for the scattering without radiation.

11.1 Vacuum Polarization



Scattering amplitude for this Feynman diagram is given by:

$$-\iota M = (-1)^1(\iota e\bar{u}_f\gamma^\nu u_i)(\frac{\iota g_{\nu\nu'}}{k^2})\int \frac{d^dq}{(2\pi)^d}[(\iota\gamma^\nu)_{\alpha\beta}\frac{\iota(\not{q}+m)_{\beta\lambda}}{q^2-m^2}(\iota e\gamma^{\mu'})_{\lambda\tau}\frac{\iota(\not{k}+\not{q}+m)_{\tau\alpha}}{(k+q)^2-m^2}]* \tag{11.1}$$

$$(\frac{-\iota g_{\mu\mu'}}{k^2})(\iota e\bar{u}_f\gamma^\mu u_i)$$

$$\frac{-\iota g_{\mu\nu}}{k^2} \to (\frac{-\iota g_{\mu\mu'}}{k^2})\Pi^{\mu'\nu'}(\frac{-\iota g_{nu\nu'}}{k^2})$$

$$\to (\frac{-\iota}{k^2})\Pi_{\mu\nu}(\frac{-\iota}{k^2})$$

Where,

$$\iota\Pi_{\mu\nu}(k) = (-\iota e)^2(-1)\int \frac{d^dq}{(2\pi)^d}tr[\gamma^\mu\frac{\iota(\not{q}+m)}{q^2-m^2}\gamma^\nu\frac{\iota(\not{q}+\not{k}+m)}{(q+k)^2-m^2}] \tag{11.2}$$

Trace Theorems and Properties of γ-Matrices

Dirac γ-matrices satisfy the commutation relation:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

Using the notation: $\not{a} = \gamma_\mu a^\mu$.

The trace theorems are: Trace of an odd number of $\gamma'_\mu s$ vanishes

$$Tr(1) = 4 \tag{11.3}$$
$$Tr(\not{a}\not{b}) = 4a.b \tag{11.4}$$
$$Tr(\not{a}\not{b}\not{c}) = 0 \tag{11.5}$$
$$Tr(\not{a}\not{b}\not{c}\not{d}) = 4[(a.b)(c.d) - (a.c)(b.d) + (a.d)(b.c)] \tag{11.6}$$
$$Tr(\gamma_5) = 0 \tag{11.7}$$
$$Tr(\gamma_5\not{a}\not{b}) = 0 \tag{11.8}$$
$$Tr(\gamma_5\not{a}\not{b}\not{c}\not{d}) = 4\iota\epsilon_{\mu\nu\lambda\sigma}a^\mu b^\nu c^\lambda d^\sigma \tag{11.9}$$

where $\epsilon\mu\nu\lambda\sigma = +1(-1)$ for μ, ν, λ, σ an even (odd) permutation of $0, 1, 2, 3$; and 0 if the two indices are the same.

Feynman Parameter Integrals

The general identity is:

$$\frac{1}{\Pi_i D_i} = (\prod_i \int_0^1 dx_i)\frac{(n-1)!\delta(1-\sum_i)}{(\sum_i x_i D_i)^n} \tag{11.10}$$

where i runs from 1 to n.
For $n = 1$

$$\frac{1}{D_1 D_2} = \int_0^1 dx \frac{1}{[xD_1 + (1-x)D_2]^2}$$

$$\frac{1}{D_1 D_2 D_3} = \int_0^1 dx \int_0^{1-x} dy \frac{2}{[xD_1 + yD_2 + (1-x-y)D_3]^3}$$

Another way the Feynman parameter integral can be defined, as an example of two factor denominator:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2} = \int_0^1 dxdy\delta(x+y-1)\frac{1}{[xA + yB]^2} \tag{11.11}$$

As an example of its use might look like:

$$\begin{aligned} \frac{1}{(k-p)^2(k^2-m^2)} &= \int_0^1 dxdy\delta(x+y-1)\frac{1}{[x(k-p) + y(k^2-m^2)]^2} \\ &= \int_0^1 dxdy\delta(x+y-1)\frac{1}{[k^2 - 2xk.p + xp^2 - ym^2]^2} \end{aligned}$$

On differentiating $eq^n(1.11)$ with respect to B, we will have,

$$\frac{-1}{AB^2} = \int_0^1 dxdy\delta(x+y-1)\frac{2y}{[xA + yB]^3}$$

On repeated differentiation with respect to B, we will have,

$$\frac{1}{AB^n} = \int_0^1 dxdy\delta(x+y-1)\frac{ny^{(n-1)}}{[xA+yB]^{n+1}}$$

The most general formula is,

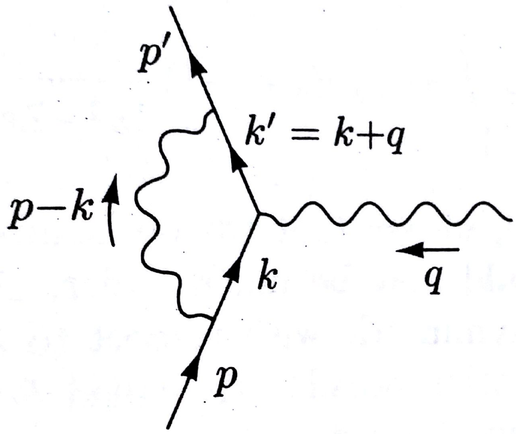
$$\frac{1}{[A_1A_2A_3...A_n]} = \int_0^1 dx_1dx_2...dx_n\delta(\sum x_i - 1)\frac{(n-1)!}{[x_1A_1+x_2A_2+...+x_nA_n]^n} \qquad (11.12)$$

The general formula can be derived by induction.
By the repeated differentiation of $eq^n(1.12)$, we can derive even more general identity:

$$\frac{1}{A_1^{m_1}A_2^{m_2}...A_n^{m_n}} = \int_0^1 dx_1dx_2...dx_n\delta(\sum x_i - 1)\frac{\Pi x_i^{m_i-1}}{[\sum x_iA_i]^{\Sigma m_i}}\frac{\Gamma(m_1+m_2+..+m_n)}{\Gamma(m_1)\Gamma(m_2)..\Gamma(m_n)} \qquad (11.13)$$

This formula is true even the m_i are not integers.

11.2 One-loop Vertex Correction



On applying the Feynman rules, we find, to the order α that

$$\Gamma^\mu = \gamma^\mu + \delta\Gamma^\mu$$

where,

$$\begin{aligned}\bar{u}(p')\delta\Gamma(p,p')u(p) &= \int\frac{d^4k}{(2\pi)^4}\frac{-\iota g_{\nu\rho}}{(k-p)^2+\iota\epsilon}\bar{u}(p')(-\iota e\gamma^\nu)\frac{\iota(\not{k}'+m)}{k'^2-m^2+\iota\epsilon}\gamma^\mu\frac{\iota(\not{k}+m)}{k^2-m^2+\iota\epsilon}(-\iota e\gamma^\rho) \\ &= 2\iota e^2\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}(p')[\not{k}\gamma^\mu\not{k}'+m^2\gamma^\mu-2m(k+k')^\mu]}{((k-p)^2+\iota\epsilon)(k'^2-m^2+\iota\epsilon)(k^2-m^2+\iota\epsilon)}\end{aligned}$$

by using an identity:

$$\gamma^\nu\gamma^\mu\gamma_\nu = -2\gamma^\mu$$

Note: the term $\iota\epsilon$ in the denominator can not be removed, as they are necessary for proper evaluation of the loop momentum integral.
Now, lets apply the Feynman parameter integral to the denominator;

$$\frac{1}{((k-p)^2+\iota\epsilon)(k'^2-m^2+\iota\epsilon)(k^2-m^2+\iota\epsilon)} = \int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{D^3} \quad (11.14)$$

where the new denominator D is;

$$D = x(k^2-m^2) + y(k'^2-m^2) + z(k-p)^2 + (x+y+z)\iota\epsilon$$

now, on expanding and using $x+y+z=1$ and $k' = k+q$,we have,

$$D = k^2 + 2k.(yq-zp) + yq^2 + zp^2 - (x+y)m^2 + \iota\epsilon$$

on shifting k to l, where,

$$l \equiv k + yq - zp$$

then,

$$D = l^2 - \Delta + \iota\epsilon$$

, where,

$$\Delta \equiv -xyq^2 + (1-z)m^2$$

Since, $q^2 < 0$ for a scattering process, Δ is $+iv$ then we can think of it as an effective mass term.

The numerator of the $eq^n(1.15)$ in terms of l. Since, D depends only on the magnitude of l, then,

$$\int \frac{d^4l}{(2\pi)^4}\frac{l^\mu}{D^3} = 0$$

$$\int \frac{D^4l}{(2\pi)^4}\frac{l^\mu l^\nu}{D^3} = \int \frac{d^4l}{(2\pi)^4}\frac{\frac{1}{4}g^{\mu\nu}l^2}{D^3}$$

The $1st$ identity follows from the symmetry. For the $2nd$ identity, note that the integral vanishes by symmetry unless $\mu = \nu$. The Lorentz invariance therefore requires that we get something proportional to $g^{\mu\nu}$.

From these identities, we have the numerator of $eq^n(1.15)$ as;

$$\begin{aligned} Numerator &= \bar{u}(p')[\not{k}\gamma^\mu\not{k}' + m^2\gamma^\mu - 2m(k+k')]u(p) \\ &= \bar{u}(p')[-\frac{1}{2}\gamma^\mu l^2 + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) \\ &\quad +m^2\gamma^\mu - 2m((1-2y)q^\mu + 2zp^\mu)]u(p) \end{aligned}$$

Now, we need to rearrange the numerator to get some useful form. we want to group everything into two groups, proportional to γ^μ and $\iota\sigma^{\mu\nu}q_\nu$.
The most easy way to do this is to aim instead for an expression of the form;

$$\gamma^\mu.A + (p'^\mu + p^\mu).B + q^\mu.C$$

we can attain this form by using only anticommutation relations like;

$$\not{p}\gamma^\mu + \gamma^\mu \not{p} = 2p^\mu$$

and the *Dirac equation*;

$$\not{p}u(p) = mu(p) \qquad and \qquad \bar{u}(p')\not{p}' = m\bar{u}(p')$$

$$\implies \bar{u}(p')\not{q}u(p) = 0$$

and $x + y + z = 1$.
Then, we have the numerator in the form;

$$\begin{aligned} Numerator &= \bar{u}(p')[\gamma^\mu.(-1/2l^2 + (1-x)(1-y)q^2 + (1-2z-z^2)m^2) \\ &+(p'^\mu + p^\mu).mz(z-1) + q^\mu.m(z-2)(x-y)]u(p) \end{aligned}$$

According to **Ward identity**, the coefficients of γ^μ must vanish in the above numerator form. This can be seen from the equations of denominator:

$$D = l^2 - \Delta + \iota\epsilon$$

where

$$\Delta = -xyq^2 + (1-z)^2 m^2$$

so the denominator is symmetric under $x \rightleftarrows y$ and the coefficient of q^μ is odd under $x \rightleftarrows y$. So, therefore vanishes when integrated over x and y.
We can eliminate $(p'^\mu + p^\mu)$ therm in the numerator by using *Gordon identity*;

$$\bar{u})(p')\gamma^\mu u(p) = \bar{u}(p')[\frac{p'^\mu + p^\mu}{2m} + \frac{\iota\sigma^{\mu\nu}q_\nu}{2m}]u(p)$$

so the final expression for the numerator is ;

$$\begin{aligned} Numerator &= \bar{u}(P')[\bar{u}(p')[\gamma^\mu.(-1/2l^2 + (1-x)(1-y)q^2 + (1-4z-z^2)m^2) \\ &+\frac{\iota\sigma^{\mu\nu}q^\nu}{2m}(2m^2 z(1-z))]u(p) \end{aligned}$$

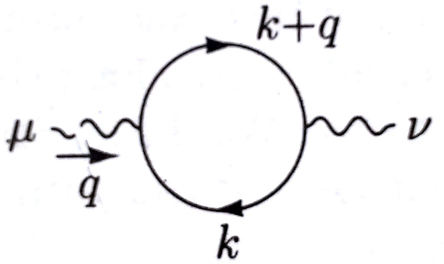
Hence, the entire expression for the $O(\alpha)$ contribution to the electron vertex then, become;

$$\begin{aligned} \bar{u}(p')\delta\Gamma(p,p')u(p) &= 2\iota e^2 \int \frac{d^4 l}{(2\pi)^4} \int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{D^3}\bar{u}(p')[\gamma^\mu.(-1/2l^2 \\ &+(1-x)(1-y)q^2 + (1-4z-z^2)m^2) + \frac{\iota\sigma^{\mu\nu}q^\nu}{2m}(2m^2 z(1-z))]u(p) \end{aligned}$$

Chapter 12

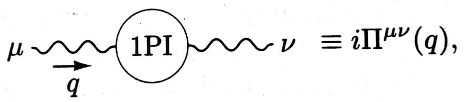
Renormalization of Electric Charge

As we have discussed before, this is the α-order *vacuum polarization* Feynman diagram, also known as *photon self energy*. This diagram (due to a loop) will change the effective field $A^\mu(x)$ seen by the scattered electron. It can shift the overall strength of this field. The interesting part is the electron-positron loop;



$$\iota\Pi_2^{\mu\nu}(q) = (-\iota e)^2(-1)\int \frac{d^4k}{(2\pi)^4} tr[\gamma_\mu \frac{\iota}{\not{k} - m}\gamma_\nu \frac{\iota}{\not{k} - \not{q} - m}] \tag{12.1}$$

Let in general, $\iota\Pi^{\mu\nu}(q)$ is defined to be the sum of all $1-particle-irreducible$insertion into the photon propagator;



So that $\Pi_2^{\mu\nu}$ is the 2nd order(in e) contribution to $\Pi^{\mu\nu}$.

$$\iota\Pi_2^{\mu\nu}(q) = (-\iota e)^2(-1)\int \frac{d^4k}{(2\pi)^4} tr[\gamma^\mu \frac{\iota(\not{k} + m)}{k^2 - m^2}\gamma^\nu \frac{\iota(\not{q} + \not{k} + m)}{(q+k)^2 - m^2}] \tag{12.2}$$

From the trace theorems and the γ-matrix properties;

$$\iota\Pi_2^{\mu\nu}(q) = -4e^2\int \frac{d^4k}{(2\pi)^4} \frac{k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}(k.(k+q) - m^2)}{(k^2 - m^2)((k+q)^2 - m^2)} \tag{12.3}$$

Now, using the Feynman parameter integral;

$$\frac{1}{(k^2 - m^2)((k+q)^2 - m^2)} = \int_0^1 dx \frac{1}{[l^2 + x(1-x)q^2 - m^2]^2} \tag{12.4}$$

where $l = k + xq$,
In terms of l the numerator of the above equation for $\Pi_2^{\mu\nu}$ will be;

$$Numerator = 2l^\mu l^\nu - g^{\mu\nu}l^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}(m^2 + x(1-x)q^2) + (terms \quad linear \quad in \quad l) \tag{12.5}$$

On wick rotation and substituting $l = \iota l_E$, we obtain

$$\iota\Pi_2^{\mu\nu}(q) \quad = \quad -4\iota e^2 \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} * \tag{12.6}$$

$$\frac{-1/2 g^{\mu\nu} l_E^2 + g^{\mu\nu} l_E^2 - 2x(1-x)q^\mu Q^\nu + g^{\mu\nu}(m^2 + x(1-x)q^2)}{(l_E^2 - \Delta)^2} \tag{12.7}$$

where $\Delta = m^2 - x(1-x)q^2$,
Since, the only tensors that appears in $\Pi^{\mu\nu}(q)$ are $g^{\mu\nu}$ and $q^\mu q^\nu$.
The ward identity tells us that

$$q_\mu = 0$$

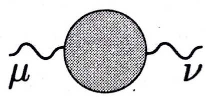
. From this, we can expect that:

$$\Pi^{\mu\nu}(q) \propto (g^{\mu\nu} - q^\mu q^\nu / q^2)$$

And we can also expect that $\Pi^{\mu\nu}(q)$ will not have a pole at $q^2 = 0$. Since, the only source of such a pole would be a single-massless-particle intermediate state, which can not occur in any $1PI$ diagram.
Note: This can be shown mathematically by the dimensional regularization. In dimensional regularization of Π_2, we find there is no such singularity in Π_2 due to a pair of massless fermions in 4-dimension.
So,we can write;

$$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2) \tag{12.8}$$

where $\Pi(q^2)$ is regular at $q^2 = 0$.
Using this notation, the exact photon two-point function is:



$$= \frac{-\iota g_{\mu\nu}}{q^2} + \frac{-\iota g_{\mu\rho}}{q^2}[(\iota q^2 g^{\rho\sigma} - q^\rho q^\sigma)\Pi(q^2)]\frac{-\iota g_{\sigma\nu}}{q^2} + \ldots \tag{12.9}$$

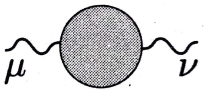
$$= \frac{-\iota g_{\mu\nu}}{q^2} + \frac{-\iota g_{\mu\rho}}{q^2}\Delta^\rho_\nu \Pi(q^2)_{\frac{-\iota g_{\mu\rho}}{q^2}} \Delta^\rho_\sigma \Delta^\sigma_\nu \Pi^2(q^2) + \ldots \tag{12.10}$$

where $\Delta^\rho_\nu = \delta^\rho_\nu - q^\rho q_\nu / q^2$.
On simplification, the expression become:

$$\equiv \frac{\iota g_{\mu\nu}}{q^2} + \frac{\iota g_{\mu\rho}}{q^2}(\delta^\rho_\nu - \frac{q^\rho q_\nu}{q^2}) \tag{12.11}$$

$$= \frac{\iota}{q^2(1 - \Pi(q^2))}(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) + \frac{\iota}{q^2}(q_\mu q_\nu) \tag{12.12}$$

Now, since, in scattering calculations, at least one end of this exact propagator will connect to a fermion line, according to Ward identity, that terms proportional to q_μ vanishes.
So, for calculating scattering amplitude(then cross section), we can write; the whole above expression reduced to:



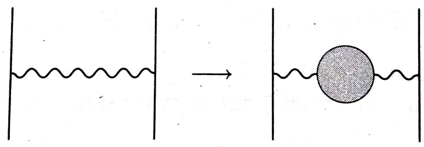
$$= \frac{-\iota g_{\mu\nu}}{q^2(1-\Pi(q^2))}$$

Now, as long as $\Pi(q^2)$ is regular at $q^2 = 0$, the exact propagator always has a pole at $q^2 = 0$. It means the photon remains massless at all orders of perturbation theory.

The residue of the $q^2 = 0$ pole is;

$$\frac{1}{1-\Pi(0)} \equiv Z \tag{12.13}$$

so, the amplitude for any low-q^2 scattering process will be shifted by this factor, related to the tree-level approximation:



$$\frac{e^2 g_{\mu\nu}}{q^2} \quad \rightarrow \quad \frac{Ze^2 g_{\mu\nu}}{q^2}$$

Since, a factor of e lies at each end of the photon propagator, we can then refer this shift by making the replacement;

$$e \quad \rightarrow \quad \sqrt{Z}e$$

This is called **Charge renormalization**.
This $\sqrt{Z}e$ is the charge that we measure in the experiment.

Chapter 13

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